notes on generalized geometry - WS26

Eugenia Boffo - Charles University Prague

January 2026

Contents

1 Algebroids

Prerequisites: vector bundles, sections, tensors, Cartan formulas, sequences, classical field theory and conserved currents.

1.1 2-dim variational problem

Some motivation. In string theory there are (conformal) surfaces embedded in 26 (10) dimensional spacetime M. If g is a metric and B a 2-form on M, the kinetic term for a string propagating on M is

$$\int_\Sigma g_{\mu\nu}(X)\,\mathrm{d}X^{\mu a}\wedge(*_h\mathrm{d}X^\nu)_a+B_{\mu\nu}(X)\,\mathrm{d}X^{\mu a}\wedge\mathrm{d}X^\nu_a\,,\qquad X:\Sigma\to M \tag{1}$$

where Latin indices in the second term are contracted with the pseudotensor ϵ_{ab} of Σ. Another equivalent expression is

$$\int_\Sigma \mathrm{dvol}\,X^*(g)+\int_\Sigma X^*(B), \tag{2}$$

The string dynamics can be found as a stationary (minimum) point of the functional. In other words, it is a minimal surface embedded in M.

For a well-posed variational problem, by Nöther theorem we do not only ask for

$$\mathcal{L}_Y g=0,\quad Y\in\mathfrak{X}^1(M) \tag{3}$$

but also

$$\mathcal{L}_Y B=\mathrm{d}\alpha\,,\quad \alpha\in\Omega^1(M)\,, \tag{4}$$

since action functionals that differ by a total derivative term (exact form) are equivalent. So we have to consider vector fields and 1-forms on the same footing. They are the sections of a sum bundle, $TM\oplus T^*M$ over M:

$$Y+\alpha\in\Gamma(TM\oplus T^*M). \tag{5}$$

Moreover, composing two infinitesimal transformations fixes a bracket

$$[Y+\alpha,Z+\beta]=\mathcal{L}_Y(Z+\beta)-\iota_Z\mathrm{d}\alpha\,, \tag{6}$$

which is known as the **Dorfman bracket** of the **Courant algebroid** $E\cong TM\oplus T^*M$. These considerations were first put forth by Ševera in his letters to Weinstein [4].

Algebroids. Let us take a step back: we have been assigning the structure of an algebra to the sections of a bundle. There are more ways of doing so:

- *Lie algebroids*: $(E\xrightarrow{\pi}M,[-,-],\rho)$ with
  - ρ is the bundle map $\rho:E\to TM$ called the *anchor*;
  - $[-,-]:\Gamma(E)\times\Gamma(E)\to\Gamma(E)$ is a Lie bracket, so it is $\mathbb{R}$-linear, skew-symmetric and has the Jacobi identity:

    $$[U,fV]=\rho(U)f\,V+f[U,V]\,,\quad [U,V]=-[V,U],\quad [U,[V,W]]+[V,[W,U]]+[W,[U,V]]=0 \tag{7}$$

    example: $E=T^*M$, $\rho=\Pi:T^*M\to TM$, $[\alpha,\beta]=\mathcal{L}_{\pi(\alpha)}\beta-\mathcal{L}_{\pi(\beta)}\alpha$
- *Leibniz algebroids*: $(E\xrightarrow{\pi}M,[-,-],\rho)$ with
  - ρ is the bundle map $\rho:E\to TM$;
  - $[-,-]:\Gamma(E)\times\Gamma(E)\to\Gamma(E)$ is neither skew nor symmetric, has the Jacobi identity (in Leibniz form, see later (9)) and it is $\mathbb{R}$-linear in the second entry:

    $$[U,fV]=\rho(U)f\,V+f[U,V]\,. \tag{8}$$

- *Courant algebroids*: $(E \xrightarrow{\pi} M, [-,-], \rho, \langle -,- \rangle)$ with
  - ρ is the (surjective) bundle map $\rho : E \to TM$;
  - $\langle -,- \rangle : \Gamma(E) \vee \Gamma(E) \to C^\infty(M)$ is a symmetric bilinear pairing;
  - $[-,-] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ is a bracket with the properties:
    1. $\mathbb{R}$-linearity in the second entry: $[U, fV] = \rho(U)f\, V + f[U,V]$;
    2. the pairing is invariant under the adjoint action: $\rho(W)\langle U,V\rangle = \langle [W,U],V\rangle + \langle U,[W,V]\rangle$;
    3. it has a symmetric part: $2\langle [U,U],V\rangle = \rho(V)\langle U,U\rangle$;
    4. Jacobi identity in Leibniz form:

$$[U,[V,W]] = [[U,W],V] + [W,[U,V]]. \tag{9}$$

  Exercise: find $[fU,V]$.
  Exercise: verify that the Dorfman bracket (6) satisfies these axioms.
  Exercise: with reference to $TM \oplus T^*M$ with the projector to TM as anchor, correct the axioms when the Courant bracket, a totally antisymmetric bracket defined as:

$$[Y+\eta, Z+\xi]_C := \mathcal{L}_Y Z + \mathcal{L}_Y \xi - \mathcal{L}_Z \eta - \frac{1}{2}\mathrm{d}(\iota_Y \xi - \iota_Z \eta) \tag{10}$$

  replaces the CA bracket. In this case, we will violate the Jacobi identity.

A neat review on algebroids is Vysoký's Ph.D. thesis, [6].

More detailed demonstration that Courant algebroids appear in the 2-dimensional variational problem. We could now go over the argument that the gauge symmetries of the string sigma model fit into a Courant algebroid. This explanation rephrases [10].

First, closed strings with the geometry of a cylinder $S^1 \times \mathbb{R}$ have a phase space T^*LM, where $LM = \{X : S^1 \to M\}$ is the *loop space*, whose canonical symplectic structure is

$$\omega = \int_{S^1} \mathrm{d}\sigma\, \delta X^\mu \wedge \delta P_\mu \quad \Leftrightarrow \quad \{X^\mu(\sigma), P_\nu(\sigma')\} = \delta^\mu{}_\nu \delta(\sigma - \sigma'). \tag{11}$$

Therefore the computation of Nöther charges when g has a Killing vector ($\mathcal{L}_v g = 0$) and $\mathcal{L}_v B = \mathrm{d}\alpha$, gives us the current (a current for a $d = 2$-dimensional sigma model, is by Nöther theorem the integral of a $d-1$ form–boundary term–and the variation under a symmetry $\frac{\delta S[g,B]}{\delta \partial_t X}\delta X$):

$$J_\epsilon = \int_{S^1} \mathrm{d}\sigma\, \epsilon(v^\mu(X)P_\mu + \xi_\mu(X)\partial_\sigma X^\mu) \tag{12}$$

where $\xi_\mu \partial_\sigma X^\mu$ is the boundary term i.e. $B_{\mu\nu}\,\mathrm{d}X^\mu \wedge \mathrm{d}X^\nu \sim B_{\mu\nu}\,\mathrm{d}X^\mu \wedge \mathrm{d}X^\nu + \mathrm{d}(\xi_\mu(X)\mathrm{d}X^\mu)$, v^μ is the component of the vector field that leaves g invariant, and

$$P_\mu = \frac{\delta S}{\delta \partial_t X}.$$

Let us call U the generalized vector with components $(v^\mu(X), \xi_\mu(X))$ and let V be another generalized vector. If we compute the Poisson bracket:

$$\{J_{\epsilon_1}(U), J_{\epsilon_2}(V)\} = -J_{\epsilon_1\epsilon_2}([U,V]_C) + \int_{S^1} \mathrm{d}\sigma(\epsilon_1 \partial_\sigma \epsilon_2 - \epsilon_2 \partial_\sigma \epsilon_1)\langle U,V\rangle \tag{13}$$

we find that it closes on the Courant bracket, plus an anomalous term. Associativity of the Poisson bracket (i.e. Jacobi identity) does not hold (it is said to hold up to higher terms or up to homotopy).

Remark 1) Exact, transitive CAs. In his letters to Weinstein [4], Ševera also classified exact CAs, those which fit in a short exact sequence:

$$0 \to T^*M \xrightarrow{j} E \xrightarrow{\rho} TM \to 0, \quad \mathrm{im}\, j = \ker \rho\,. \tag{14}$$

He showed that they can be classified with a closed 3-form. Exact CAs are isomorphic to the sum bundle of tangent and cotangent bundle.

Another interesting situation arises when the sequence is not exact. Then the algebroids are *transitive* Courant algebroids, with total space

$$E \cong T^*M \oplus \mathfrak{g} \oplus TM\,, \tag{15}$$

for $\mathfrak{g}$ a semisimple Lie algebra with $\langle -, - \rangle_{\mathfrak{g}}$ an invariant pairing.

Remark 2) Aut. Automorphism group of the Dorfman bracket: what leaves the bracket invariant? Certainly diffeomorphisms (Lie derivative and contractions are invariant under pullbacks), but we can show that also $B \in \Omega^2_{\mathrm{cl}}(M)$ is in Aut:

$$\mathrm{Aut}_{\mathrm{Dorf}}(E) = \Omega^2_{\mathrm{cl}}(M) \rtimes \mathrm{Diff}(M)\,. \tag{16}$$

For the purpose of the proof, let us write

$$e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}\,. \tag{17}$$

Then

$$e^{-B}[e^B U, e^B V] = e^{-B}([U,V] + \iota_X \mathrm{d}\iota_Y B + \mathrm{d}\iota_X\iota_Y B - \iota_Y \mathrm{d}\iota_X B) \tag{18}$$

$$= [U,V] - \iota_{[X,Y]}B + \iota_X \mathrm{d}\iota_Y B + \mathrm{d}\iota_X\iota_Y B - \iota_Y \mathrm{d}\iota_X B \tag{19}$$

$$= [U,V] + \iota_Y\iota_X \mathrm{d}B \tag{20}$$

and the conclusion follows.

Exercise: Try to use e^{g+B} now, with g a metric. Does it still give rise to a homomorphism? Verify the axioms, using $\langle e^{g+B}(U), e^{g+B}V\rangle = \langle U,V\rangle + g(X,Y)$.

2 Relation to QP manifolds

Motivation. QP manifolds appear in the context of topological field theories, constructed via the celebrated AKSZ algorithm [15].

In the acronym QP, Q stands for a homological vector field $Q^2 = 0$ and P is a Poisson structure (we will just consider invertible ones, i.e., symplectic manifolds). An alternative, equivalent wording is *differential graded (dg) symplectic manifolds of degree n.*

Primer of graded manifolds. To understand (positively) graded manifolds, let us start with a *graded vector space V*. It decomposes into subspaces,

$$V = \bigoplus_{i \in I \subset \mathbb{N}} V_i \tag{21}$$

and vectors forming a basis of V_i are said to be of degree i. In most applications, a $\mathbb{Z}_2$ grading is also assigned (in plain English, even degree vectors have even parity, while odd degree vectors are parity odd). A useful gadget is *the degree shift*, denoted by:

$$V_i[k] := V_{i+k}, \quad i, k \in \mathbb{N}.$$

Then the concept of a *graded vector bundle* can be suggested: similar to a vector bundle, it has as fibers the graded vectors of V.

A *graded manifold* $\mathcal{M}$ is modeled over a regular manifold M with structure sheaf

$$\mathcal{O}_{\mathcal{M}} \cong \mathcal{O}_M \otimes \Lambda^{\bullet}_V \,. \tag{22}$$

To be concrete, for the smooth functions on an open set $\mathcal{U} \subset \mathcal{M}$

$$C^{\infty}(\mathcal{U}) \cong C^{\infty}(U) \otimes \Lambda^{\bullet} V \,. \tag{23}$$

Beware that is customary to invoke the isomorphism

$$\wedge V \cong \mathrm{Sym} V[1]$$

so a function ϕ of $\mathcal{M}$ can be expressed as:

$$\phi(x, \theta_i) = \varphi(x)_J \prod_{i=1}^{i_{\max}=I} \prod_{n_i=0}^{n_{i,\max}=\dim V_i} \theta_i^{k^i_{n_i}} \qquad J = (k^1_1, \ldots, k^1_{n_{1,\max}}, k^2_1, \ldots k^2_{n_{2,\max}}, \ldots) \,. \tag{24}$$

In the much simpler situation of a $\mathbb{Z}_2$-graded manifold (supermanifold) a function Ψ is

$$\Psi(x, \theta^i) = \psi(x) + \psi(x)_i \theta^i + \psi(x)_{ij} \theta^i \theta^j + \cdots + \psi(x)_{i_1 \ldots i_{\dim V}} \theta^i \ldots \theta^{i_{\dim V}} \,.$$

An important characterization of graded manifolds is **Batchelor's theorem**:
Every supermanifold is (non-canonically) isomorphic to a graded vector bundle.
In particular, the transition functions are $x = x(x')$ and $\theta' = M\theta$ with $M \in \mathrm{GL}(\dim V)$.

Some trivia on graded symplectic geometry. Over $\mathcal{M}$ we can find a symplectic form (i.e. a non-degenerate, closed 2-form). There are some immediate consequences for symplectic geometry over a graded manifold, contrary to the standard case:

1. the 2-form is exact. This follows from the existence of a Euler vector field E and the fact that ω has a degree $n \neq 0$:
$$\mathcal{L}_E \omega = n\omega \Rightarrow \frac{1}{n} \mathrm{d}\iota_E \omega = \omega \,. \tag{25}$$

2. Suppose to have a vector field Q of degree m that preserves the symplectic form, $\mathcal{L}_Q\omega = 0$. Then, if $n \neq -m$, the vector field is Hamiltonian:

$$\iota_{[E,Q]}\omega = [\mathcal{L}_E, \iota_Q]\omega \tag{26}$$

$$m\,\iota_Q\omega = \iota_E \mathrm{d}\iota_Q\omega + \mathrm{d}\iota_E\iota_Q\omega - \iota_Q n\omega \tag{27}$$

$$\iota_Q\omega = \frac{1}{n+m}(\cancel{\iota_E\mathcal{L}_Q\omega} + \mathrm{d}\iota_E\iota_Q\omega). \tag{28}$$

QP manifolds correspond to algebroids

The correspondence between QP manifolds and algebroids appeared in [8] but the work is attributed jointly to Ševera–Roytenberg.

Equivalence of Lie algebroid with $T^*[1]M$. On $T^*[1]M$ with coordinates x and η,[1] we have $Q = \eta\partial_x$ and $\omega = \mathrm{d}x \wedge \mathrm{d}\eta$. They both have degree 1. The Poisson bracket of two functions of $T^*[1]M$ corresponds to the Schouten bracket of multivector fields. Let us check it over degree 1 functions:

$$\{w(x)^i\eta_i, u(x)^k\eta_k\} = \left(w(x)^i\partial_i u(x)_k - u(x)^i\partial_i w(x)_k\right)\eta^k \;\Leftrightarrow\; [W, U] \tag{29}$$

$$\{w(x)^i\eta_i, f(x)\} = w(x)^i\partial_i f(x) \;\Leftrightarrow\; [W, f] = \rho(W)(f) \equiv W(f) \tag{30}$$

Of course the bracket of the wannabe algebroid satisfies Jacobi identity (because the Poisson bracket does). So $T^*[1]M$ with $Q = \eta\partial_x$ and the canonical symplectic form is the Lie algebroid $(\Lambda^\bullet(TM), [-,-]_{SN})$.

Poisson sigma model: Let A be a 1-form on M with values in T^*M, and X maps $\Sigma \to M$. P^{ij} is the i,j-component of a Poisson tensor, which by definition satisfies

$$[P, P] = 0\,. \tag{31}$$

Then the Poisson sigma model has the classical action

$$\int_\Sigma A_i \wedge \mathrm{d}X^i + P^{ij}A_i \wedge A_j\,, \tag{32}$$

and the invariance:

$$\delta_\epsilon X^j = \epsilon_i P^{ij}, \quad \delta_\epsilon A_i = \mathrm{d}\epsilon_i + P^{jk}{}_{,i}A_j\epsilon_k \tag{33}$$

(the structure constants are derivatives of the Poisson bivector). Note: if we let the form take values in $\mathfrak{su}(n)$, the AKSZion (32) is a topological term of YM in two dimensions in first order formulation. Cattaneo and Felder showed that a Feynman diagram expansion of the above AKSZion gives Kontsevich's deformation quantization.

Equivalence of CAs with the dg symplectic graded manifolds of degree 2. The graded manifold under consideration is $T^*[2]T[1]M$, with coordinates

$$\begin{array}{ccccc} \text{coord.} & x^i & \theta^i & \chi_i & p_i \\ \text{space} & M & T[1] & T^*[2]T[1] & T^*[2]M \\ \text{degree} & 0 & 1 & 1 & 2 \end{array} \tag{34}$$

Then the canonical symplectic structure is $\omega = \mathrm{d}x \wedge \mathrm{d}p + \mathrm{d}\theta \wedge \mathrm{d}\chi$. Note that due to their Grassmann parity, $\mathrm{d}\theta \wedge \mathrm{d}\chi = \mathrm{d}\chi \wedge \mathrm{d}\theta$, i.e. a symplectic structure over a field of characteristic 2 is a metric. Then the Hamiltonian (equivalent to assigning a homological vector field under property 2.) is

$$\Theta = p_i\rho^{i\alpha}(x)\xi_\alpha + \frac{1}{3!}C^{\alpha\beta\gamma}(x)\xi_\alpha\xi_\beta\xi_\gamma\,, \quad \xi_\alpha \equiv (\theta^i, \chi_i) \tag{35}$$

[1] Note that $C^\infty(T^*[1]M) \cong \mathfrak{X}(M)$

which is homogeneous of degree 3. To show that QP of degree 2 implies the structure of a Courant algebroid over $TM \oplus T^*M$, it sufficient to employ the *derived bracket* with the Hamiltonian.

$$\begin{aligned} \{\{U^\alpha(x)\xi_\alpha, \Theta\}, V^\beta\xi_\beta\} =& \{U^\alpha p_i \rho^i{}_\alpha - \rho^{i\gamma}\partial_i U^\alpha \xi_\alpha\xi_\gamma + \frac{1}{2}U_\alpha C^{\alpha\beta\gamma}\xi_\beta\xi_\gamma, V^\beta\xi_\beta\} && (36)\\ =& U^\alpha\rho^i{}_\alpha\partial_i V^\beta - V_\gamma\rho^{i\gamma}\partial_i U^\alpha\xi_\alpha + \rho^{i\gamma}(\partial_i U^\alpha)V_\alpha\xi_\gamma - U_\alpha V_\beta C^{\alpha\beta\gamma}\xi_\gamma\,, && (37)\end{aligned}$$

which is a component expression for the Dorfman bracket, twisted by an antisymmetric 3-tensor $C^{\alpha\beta\gamma}$. The tensor $\rho^{i\alpha} \in \mathrm{Hom}(E, TM)$ is the anchor map, as evident from

$$\{\{U^\alpha(x)\xi_\alpha, \Theta\}, f(x)\} = \{U^\alpha p_i\rho^i{}_\alpha + \ldots, f(x)\} = U^\alpha\rho^i{}_\alpha\partial_i f(x)\,. \tag{38}$$

The pairing of the Courant algebroid is due to the metric on the degree 1 coordinates. Consistency (i.e. that the bracket, anchor and pairing respect the axioms) follows from nilpotency of the Hamiltonian vector field, or equivalently:

$$\{\Theta, \Theta\} = 0 \tag{39}$$

which is non-trivial in the realm of graded geometry.
In conclusion, $T^*[2]T[1]M$ with Θ given above and the canonical symplectic structure is equivalent to $(TM \oplus T^*M, \rho, [-,-]_{\mathrm{Dorfman}}, \langle -,-\rangle)$.

Courant sigma model: example of a BV (AKSZ) theory because it is over vector spaces of the kind $V \oplus \Pi V^*$ where $V = T[1]M$. We shall see just the classical part of the AKSZion. Be N a 3-dimensional manifold, and promote the coordinates of $T^*[2]T[1]M$ to maps $N \to T^*[2]T[1]M$. Then

$$\int_N p_i\,\mathrm{d}x^i + \frac{1}{2}\xi^\alpha\, g_{\alpha\beta}\,\mathrm{d}\xi^\beta - \Theta \tag{40}$$

is the classical Courant sigma model. Its quantization is easily produced by means of the AKSZ algorithm.

3 Geometries

Generalized geometry encompasses Riemannian, complex and symplectic geometry, as Hitchin, Gaultieri and Cavalcanti made us aware of, in [1]. These geometries are found when scanning what the *generalized complex structures* of (the complexification of) E are. Indeed, replicating the standard case, a generalized complex structure is

$$\mathcal{J} \in \mathrm{End}(E), \qquad \mathcal{J}^2 = -1, \quad \mathcal{J}^T = -\mathcal{J}.$$

The first condition defines an almost integrable complex structure but the second one is unexpected from the viewpoint of standard differential geometry and is an antisymmetry condition.

Complex and symplectic geometry. Let us decompose $\mathcal{J}$ in blocks, adapted to the decomposition $E \cong TM \oplus T^*M$. Then it does not take long to convince ourselves that we can use a complex structure $J \in \mathrm{End}(TM)$ ($J^2 = -1$ and integrable, $[X,Y] + J([JX,Y] + [X,JY]) - [JX,JY] = 0$), and $\omega \in \Lambda^2 T^*M$ with $\mathrm{d}\omega = 0$ to build the generalized almost complex structures:

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}, \quad \mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}. \tag{41}$$

Note, however, that a complex and a symplectic structure are just sufficient conditions for the existence of $\mathcal{J}$: one can find examples where neither J or ω are present on M, yet there is a generalized complex structure (see *nilmanifolds* in 6 dimensions).

Regarding the integrability of the generalized almost complex structure, the definition is that the $+\mathrm{i}$-subbundle of E (hence the $-\mathrm{i}$-subbundle too) must be involutive w.r.t. the Courant bracket. This yields the closure condition for ω and the involutivity of the Lie bracket of vector fields.

Generalized Kähler structures. Notably, in complex differential geometry a *Kähler metric* can be given as

$$h := \omega(-, J), \tag{42}$$

which is a metric because it is symmetric: $h^* = \omega^*(-, J^*) = (-\omega)(-, -J) = h$). By analogy, this suggests that multiplying $\mathcal{J}_J$ and $\mathcal{J}_\omega$ is a way to obtain a *generalized Kähler metric*.

An important G-structure. Let us go back to the real case. For an exact CA, the bundle $E \xrightarrow{\pi} M$ can be seen as a principal bundle with structure group

$$O(d,d) \subset GL(2d). \tag{43}$$

Structure groups like this are called G-structures. The reduction of the structure group is determined by the pairing. $O(d,d)$ has physical interest because it is the T-duality group: an $O(d,d)$-invariant theory is manifestly invariant under T-dualities (symmetries of effective string actions).

The orthogonal group of split signature $O(d,d)$ is generated by

- B-field transform, $B \in \Omega^2(M)$: the group element is

$$\begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}, \quad \langle U + B(U), V + B(V) \rangle = \langle U, V \rangle + B(U,V) + B(V,U) = \langle U, V \rangle. \tag{44}$$

- β-transform, where β is a bivector of M: the group element is

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \tag{45}$$

  which, by a similar calculation as before, leaves the pairing invariant.

- the last group element is made up from $\mathrm{Aut}\, TM \ni R$

$$\begin{pmatrix} R & 0 \\ 0 & R^{-T} \end{pmatrix}. \tag{46}$$

  Invariance of the pairing is checked, relies on $R^{-1}R = 1$.

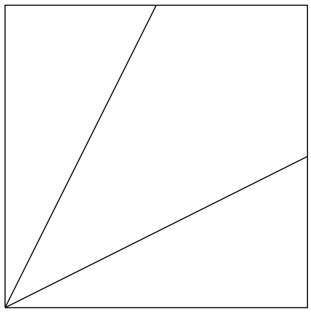

Figure 1: The generalized metric turns $TM \oplus T^*M$ (as a $1+1$-dimensional vector space, its generators are the horizontal and vertical axes) into $E_+ \oplus E_-$, another set that spans the same total vector space. Needs labels!

Generalized metric. It is possible to define a metric via an involution $\tau : E \to E$, $\tau^2 = 1$. Then the metric is

$$G \in S^2(E^*), \quad G := \langle -, \tau(-)\rangle \,. \tag{47}$$

In fact, $G(U,V) = \langle U, \tau(V)\rangle = \langle \tau(V), U\rangle \overset{\tau^2=1}{=} \langle V, \tau(U)\rangle = G(V,U)$.
An immediate consequence to observe is that E splits into two subbundles, the eigenbundles of τ: $E_+ \oplus E_-$.
Secondly, note that we can embed TM into $E_\pm$ via the graph of a Riemannian metric and a 2-form on M:

$$\mathrm{graph}(\pm g + B) : TM \to E_\pm \tag{48}$$

while the whole projector $P_\pm$ is

$$P_\pm = \frac{1}{2}(\mathrm{id} + (\pm g + B))\left([\mathrm{id} \mp g^{-1}B] \oplus \pm g^{-1}\right) : E \cong TM \oplus T^*M \to E_\pm \tag{49}$$

so $\tau = P_+ - P_-$ in matrix form:

$$\tau = \begin{pmatrix} g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} \tag{50}$$

and one can easily read off the generalized metric, $G = \langle -, \tau(-)\rangle$, by an unorthodox clockwise rotation.
An observation is immediate: *a generalized metric is uniquely determined by a metric and a 2-form.*

3.1 Physics of $N = 2$ stringy sigma models and generalized geometry

This rephrases an explanation by Zabzine [7]. In the notation of lecture 1, consider a superstring: promote $X(\sigma)$ and $P(\sigma)$ to be *superfields* of $S^{1|1}$ ($\mathbb{Z}_2$-graded fields)

$$\chi = X + \vartheta\Xi(\sigma), \quad \Pi = \pi(\sigma) + \mathrm{i}\vartheta P \,, \tag{51}$$

so that Π is Grassmann odd. The symplectic structure on the target space phase space $T^*[1]\mathbb{L}M$ (the cotangent to the super loop space) is

$$\omega = \mathrm{i}\int_{S^{1|1}} \mathrm{d}\sigma\, \mathrm{d}\vartheta\, \delta\Pi_\mu \wedge \delta\chi^\mu \tag{52}$$

These superfields should replace their counterparts to give the superstring sigma model Lagrangian that we saw in the first lecture, (1). By construction, this system has one supersymmetry charge Q_1, but Gates–Hull–Roček [14] were able to show that a second supersymmetry Q_2 is present. Their algebra is

$$\{Q_1, Q_1\} = 2\not{P} = \{Q_2, Q_2\} \tag{53}$$

but otherwise they anticommute with each other. Note that the slashed notation means $\Gamma^i \mathrm{P}_i$ where Γ is a Clifford algebra Gamma matrix $\{\Gamma^i, \Gamma^j\} = 2g^{ij}$ (for some metric g). However here the index is on the line, so $\Gamma^i_{1\times 1} = 1$.

It turns out that one can represent Q_2 as

$$Q_2 = -\frac{1}{2} \int_{S^{1|1}} \mathrm{d}\sigma \mathrm{d}\vartheta \; 2D\chi^\rho \Pi_\nu J^\nu{}_\rho + D\chi^\nu D\chi^\rho L_{\nu\rho} + \Pi_\nu \Pi_\rho K^{\nu\rho} \tag{54}$$

where $D := \partial_\vartheta + \mathrm{i}\vartheta\partial_\sigma$ is the superderivative. For the momentum of the susy algebra (53) we shall take:

$$\mathrm{P} = \int_{S^{1|1}} \mathrm{d}\sigma \mathrm{d}\vartheta \; \Pi_\nu \, \partial_\sigma \chi^\nu \, . \tag{55}$$

Notice that $(D\chi, \Pi) =: \mathbf{U}$ can be interpreted as a pullback section of $\Phi^*(T[1]M \oplus T^*[1]M)$ where $\Phi \in \mathbb{L}M$, and the tensors appearing in the realization of Q_2 are combined in

$$\mathcal{J} = \begin{pmatrix} -J & K \\ L & J^t \end{pmatrix} \tag{56}$$

so that Q_2 is written, with the help of the canonical pairing, as:

$$Q_2 = -\frac{1}{2} \int_{S^{1|1}} \mathrm{d}\sigma \mathrm{d}\vartheta \; \langle \mathbf{U}, \mathcal{J}\mathbf{U} \rangle \, . \tag{57}$$

For the susy transformations of the fields, one can read them from:

$$\delta\chi = \{Q_2, \chi\}, \quad \delta\Pi = \{Q_2, \Pi\} \, . \tag{58}$$

This supports the following conclusion: *there is $N = 2$ supersymmetry iff the target space is a generalized complex manifold.* The proof is a lengthy calculation of the susy algebra with Q_2 (53) and the Poisson bracket associated to the symplectic form (52).

From a sketch of the proof, one gets the algebraic conditions

$$K^T = -\,K \tag{59}$$

$$L^T = -\,L \tag{60}$$

$$J^2 = -\,1 \tag{61}$$

as well as some differential conditions, which amount to the integrability of the generalized complex structure, phrased in terms of the projectors $\pi_\pm = \frac{1}{2}(\mathrm{id} \pm \mathrm{i}\mathcal{J})$:

$$\pi_\mp [\pi_\pm U, \pi_\pm V]_C = 0 \, . \tag{62}$$

4 Connections and tensors

On the generalized tangent bundle, it would be interesting to have a covariant differential calculus and to be able to define torsion and curvature tensors. The datum of an algebroid comes naturally equipped with a derivation, the Lie/Leibniz/Courant bracket. Moreover, for a Courant algebroid with a skew-symmetric bracket, it is possible to extrapolate the definition of a differential operator from the axiom on the symmetric part of the bracket:

$$\mathcal{D} : C^\infty(M) \to \Gamma(E) \tag{63}$$

$$f \mapsto g_E^{-1} \circ \rho^T \circ \mathrm{d}(f) \tag{64}$$

for g_E the pairing of the CA and d the de Rham differential. But are there more such notions? This last lecture is about answering this question for CAs. Most of the familiar results of standard Riemannian geometry will slip away here.

4.1 Connections

An (affine) connection for a vector bundle $E \xrightarrow{\pi} M$ is an operator

$$\nabla : \Gamma(E) \to \Gamma(TM) \otimes \Gamma(E) \tag{65}$$

which is a derivation:

$$\nabla_X\, fU = X(f)\, U + f \nabla_X U. \tag{66}$$

Note that it can be turned into an E-on-E connection,

$$\nabla = \nabla_{\rho(-)}. \tag{67}$$

Now $\nabla : \Gamma(E) \to \Gamma(E) \otimes \Gamma(E)$.
A Courant algebroid connection is an affine connection that is also requested to be compatible with the pairing:

$$\langle \nabla U, V \rangle + \langle U, \nabla V \rangle = \rho(-)\langle U, V \rangle\,.$$

Bismut connection. The generalized metric leaves two eigenbundles defined, according to their eigenvalue w.r.t. the involution τ. Therefore we could get the projectors $P_\pm$ (49), that can also be expressed as the graph of the generalized metric:

$$P_\pm(U) = U \pm G(U)\,. \tag{68}$$

Let us now denote sections of the $E_\pm$ bundles with a $\pm$ subscript. Define the linear map ($B \in \Omega^2$):

$$\begin{aligned} C &: \Gamma(E) \to \Gamma(E) \\ X + \alpha &\mapsto C(X+\alpha) = X - \alpha + B(X). \end{aligned} \tag{69}$$

One can show that the *Bismut connection*

$$\nabla_e e' := [e_+, e'_-]_- + [e_-, e'_+]_+ + [C(e_+), e'_+]_+ + [C(e_-), e'_-]_- \tag{70}$$

is a CA connection. Furthermore, it is a metric connection w.r.t. the generalized metric G.

Another expression of the Bismut connection as a TM-on-E connection is:

$$\nabla_X = e^{-B} \begin{pmatrix} \nabla_X & -\frac{1}{2} g^{-1} H(X, g^{-1}(*), -) \\ -\frac{1}{2} H(X, *, -) & \nabla_X \end{pmatrix} e^B \tag{71}$$

We postpone results about uniqueness of the torsion-free, G-metric connections to the next subsection.

Remarks:

- for exact CA: a connection TM-on-E could be given as the splitting of the exact sequence:

$$0 \to T^*M \xrightarrow{j} E \underset{\nabla}{\overset{\rho}{\rightleftarrows}} TM \to 0 \tag{72}$$

  which means that $\rho(\nabla Y) = Y$.

- Assume that the Courant algebroid has a *"dull bracket"* [16], i.e. $[[-,-]] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ with the Leibniz property in both inputs (so it is interpolating between a Lie algebroid and a Leibniz algebroid bracket)

$$[[gU, fV]] = g\rho(U)(f)V + fg[[U,V]] - f\rho(V)(g)U.$$

  Then the difference between the CA bracket and the dull bracket is an affine CA connection [17]:

$$\langle [U,V] - [[U,V]], W\rangle = \langle \nabla_W U, V\rangle. \tag{73}$$

  One can verify that metric compatibility holds if $[[-,-]]$ is skew-symmetric.

- A cutting-edge viewpoint on connections: representations up to homotopy [18]. These are not so well studied. The setup is as follows:

  - an operator ∂ on V turning it into a complex;
  - an E-on-V connection ∇;
  - a 2-form $\omega_2 \in \Omega^2(E; \mathrm{End}(V))$ with

$$[\partial, \omega_2] + R_\nabla = 0$$

  - higher forms, recursive relations.

  It roughly means that there are higher order curvature tensors.

4.2 Tensors

Given connections on the sections of an algebroid, it is natural to ask if the notions of torsion and curvature still hold. The answer is positive for Lie algebroids, but corrections are needed in the case of CAs: Both the non-skewsymmetric bracket and its skewsymmetric version spoil the transformation property of the tensor, and of course using the former requires corrections if we want to preserve antisymmetry.

Torsion. Let $(E, [-,-], \rho, \langle -,-\rangle)$ be a CA. Let ∇ be an E-on-E connection. Then the naive torsion tensor for the connection

$$T(U,V) := \nabla_U V - \nabla_V U - [U,V] \tag{74}$$

fails to be a tensor (we are not worrying about whether it is symmetric, antisymmetric or it has no definite symmetry, for now). While

$$T(U, fV) = fT(U,V),$$

the problematic part is showing $C^\infty(M)$-linearity in the first entry. Recalling the axiom on the symmetric part of the CA bracket and $\mathbb{R}$-linearity:

$$\begin{aligned}\langle [fU, V], W\rangle &= \rho(W)\langle fU, V\rangle - \langle [V, fU], W\rangle \\ &= \rho(W)(f)\langle U, V\rangle - \rho(V)(f)\langle U, W\rangle + f\langle [U,V], W\rangle.\end{aligned} \tag{75}$$

Note that even if we were using the skew-symmetric version, $C^\infty(M)$-linearity would still be missing:

$$[fU, V]_C = -\rho(V)(f)U + \frac{1}{2}\mathcal{D}f\langle U, V\rangle + f[U, V]_C.$$

An option for a well posed torsion tensor is to correct the extra term that spoils the property of a tensor. This solution is due to Gualtieri [3] (non-skewsymmetric bracket version):

$$T(U, V, W) := \langle \nabla_U V - \nabla_V U - [U, V], W\rangle + \langle \nabla_W U, V\rangle\,. \tag{76}$$

Leveraging on compatibility of ∇ with the pairing, one can show that the torsion is totally antisymmetric:

$$T \in \Lambda^3 E^*\,, \tag{77}$$

while the torsion of standard Riemannian geometry, as a 3-tensor, is an element of $\Lambda^2(T^*M) \otimes T^*M$.

We are now ready to set up the generalized geometry definition of Levi-Civita connection and then we shall accept that the metric and torsion free connections of a Courant algebroid are highly non-unique.

Definition: Consider $V_\pm \subseteq E$, a generalized metric. The CA connection ∇ is Levi-Civita if

1. $\nabla(V_+) \subseteq V_+$,
2. $T^\nabla = 0$.

The set of Levi-Civita connections is non-empty but also does not have just one element, so there is *no fundamental theorem of Riemannian geometry* in generalized geometry [20].

Curvature. Similarly, also the naive Riemann tensor is doomed by the same destiny of not being a proper tensor. Anyway, proposals are available (see [6, sec. 2.4]) and an especially convenient one holds for E-on-E connections induced by TM-on-E connections:

$$R(U, V) := [\nabla_U, \nabla_V] - \nabla_{[U,V]} \tag{78}$$

with the non-skew version of the bracket. In fact, tensoriality fails in a controlled way:

$$R(fU, V) = fR(U, V) - \langle U, V\rangle \nabla_{\mathcal{D}f}$$

and the last term vanishes when $\nabla \equiv \nabla_\rho$, as $\rho \circ \mathcal{D} = 0$.

For applications to physics it is more relevant to provide the Ricci tensor. Its definition is based on the eigenbundles of the generalized metric and a *divergence operator* in the context of generalized geometry, which is the following derivation

$$\mathrm{div} : \Gamma(E) \to C^\infty(M) \tag{79}$$

$$\mathrm{div}(fU) = f\,\mathrm{div}\,U + \rho(U)(f)\,. \tag{80}$$

Divergences form an affine space over $\Gamma(E)$: a divergence operator can be found from the divergence corresponding to a density μ on M. A density is a volume form on the manifold, i.e. any differential form of maximal dimension transforming with the determinant of the Jacobian of the transformation. The divergence associated with μ is

$$\mathrm{div}_\mu := \mu^{-1}\,\mathcal{L}_{\rho(-)}\,\mu \tag{81}$$

so that every divergence can be obtained as

$$\mathrm{div} = \mathrm{div}_\mu + \langle e, -\rangle\,. \tag{82}$$

Very remarkably, divergences satisfy the following property (failure to be a derivation of the CA bracket, while div_μ is)

$$\mathrm{div}[U, V] - [\mathrm{div}\,U, V] + [U, \mathrm{div}V] = \langle [e, U], V\rangle\,, \tag{83}$$

for some section e.

Eventually, the Ricci tensor of generalized geometry is defined in [9] for an E_+ subbundle and a divergence div:

$$\mathrm{GRic}_{E_+,\mathrm{div}}(U_+, V_-) := \mathrm{div}\,[V_-, U_+]_+ - [V_-, \mathrm{div}\, U_+] - \mathrm{Tr}_{E_+}[[-, V_-]_-, U_+]_+ \tag{84}$$

which can be shown to be an $E_+^* \otimes E_-^*$ antisymmetric tensor.

Why is this physically meaningful? When we have an exact CA with a given closed 3-form H and a generalized metric G (that ultimately corresponds to the Riemannian metric $g \in \vee^2 T^*M$), and the divergence is $\mathrm{div} = \mathrm{Tr}\,\nabla^H \circ \rho$ (∇^H the metric connection with H-torsion), then

$$\mathrm{GRic}_{g,H,\mathrm{div}}(U_+, V_-) = \left(R[g]_{ij} + \frac{1}{6} H_{imk} H_j{}^{mk} \right) \rho(U_+)^i \, \rho(V_-)^j \tag{85}$$

i.e. the generalized Ricci curvature is the Ricci tensor of the connection ∇^H applied to the vector fields corresponding to $\rho(U_+)$ and $\rho(V_-)$.

Remarks:

- Of course this is not even remotely close to full 10 dimensional Supergravity, whose field content consists of dilaton, metric, H-form, Ramond-Ramond fluxes (higher forms satisfying a higher version of Maxwell theory), as well as the fermionic partners like dilatino, gravitino etc.. Bosons have been understood since the seminal work [19], but fermions have proven to be more elusive. However, the formalism of generalized geometry stretches also to the fermions, see the recent progress of [5] and subsequent articles, which recast and tidy up Supergravity as well as setting the stage for quantization.

- The results presented so far are just valid in first order of perturbation theory in the string coupling constant α', so corrections/re-summing higher order in α' is still an open question.

References

[1] M. Gualtieri, "Generalized complex geometry," [arXiv:math/0401221 [math.DG]].

[2] M. Gualtieri, "Generalized Kahler geometry," [arXiv:1007.3485 [math.DG]].

[3] M. Gualtieri, "Branes on Poisson varieties," doi:10.1093/acprof:oso/9780199534920.003.0018 [arXiv:0710.2719 [math.DG]].

[4] P. Ševera, "Letters to Alan Weinstein about Courant algebroids," [arXiv:1707.00265 [math.DG]].

[5] J. Kupka, C. Strickland-Constable and F. Valach, "Batalin-Vilkovisky Formulation of N=1 Supergravity in Ten Dimensions," Phys. Rev. Lett. **134** (2025) no.21, 211602 doi:10.1103/PhysRevLett.134.211602 [arXiv:2501.18008 [hep-th]].

[6] J. Vysoky, "Geometry of Membrane Sigma Models," [arXiv:1512.08156 [math.DG]].

[7] M. Zabzine, "Lectures on generalized complex geometry and supersymmetry," Archivum Math. **42** (2006) no.5, 119-146 [arXiv:hep-th/0605148 [hep-th]].

[8] D. Roytenberg, "On the structure of graded symplectic supermanifolds and Courant algebroids," [arXiv:math/0203110 [math.SG]].

[9] P. Ševera and F. Valach, "Courant Algebroids, Poisson–Lie T-Duality, and Type II Supergravities," Commun. Math. Phys. **375** (2020) no.1, 307-344 doi:10.1007/s00220-020-03736-x [arXiv:1810.07763 [math.DG]].

[10] A. Alekseev and T. Strobl, "Current algebras and differential geometry," JHEP **03** (2005), 035 doi:10.1088/1126-6708/2005/03/035 [arXiv:hep-th/0410183 [hep-th]].

[11] P. Schaller and T. Strobl, "A Brief introduction to Poisson sigma models," Lect. Notes Phys. **469** (1996), 321-333 doi:10.1007/BFb0102573 [arXiv:hep-th/9507020 [hep-th]].

[12] R. Smith, https://tracingcurves.wordpress.com/tag/generalised-geometry/

[13] U. Lindström, R. Minasian, A. Tomasiello, M. Zabzine, "Generalized Complex Manifolds and Supersymmetry," Comm. Math. Phys. **257** (2005), 235-256 doi:10.1007/s00220-004-1265-6 [arXiv:hep-th/0405085 [hep-th]]

[14] S. J. Gates, Jr., C. M. Hull and M. Rocek, "Twisted Multiplets and New Supersymmetric Non-linear Sigma Models," Nucl. Phys. B **248** (1984), 157-186 doi:10.1016/0550-3213(84)90592-3

[15] M. Alexandrov, A. Schwarz, O. Zaboronsky and M. Kontsevich, "The Geometry of the master equation and topological quantum field theory," Int. J. Mod. Phys. A **12** (1997), 1405-1429 doi:10.1142/S0217751X97001031 [arXiv:hep-th/9502010 [hep-th]].

[16] M. Jotz Lean, "Dorfman connections and Courant algebroids," 2015, https://arxiv.org/abs/1209.6077.

[17] E. Boffo and P. Schupp, "Deformed graded Poisson structures, Generalized Geometry and Supergravity," JHEP **01** (2020), 007 doi:10.1007/JHEP01(2020)007 [arXiv:1903.09112 [hep-th]].

[18] A. Chatzistavrakidis, "Basic curvature tensors," talk at Corfu Summer Institute, 2025, https://www.physics.ntua.gr/corfu2025/Talks/a_chatzistavrakidis@gmail_com.01.pdf

[19] A. Coimbra, C. Strickland-Constable and D. Waldram, "Supergravity as Generalised Geometry I: Type II Theories," JHEP **11** (2011), 091 doi:10.1007/JHEP11(2011)091 [arXiv:1107.1733 [hep-th]].

[20] M. Garcia-Fernandez, "Ricci flow, Killing spinors, and T-duality in generalized geometry," Advances in Mathematics **350**, (2019) doi :10.1016/j.aim.2019.04.038 [arXiv:1611.08926 [math.DG]]